Kirchhoff's Circuit Laws

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Abstract

Simple electrical circuits which constitute a closed-loop are subject to linear analysis. The characteristics of electric circuits relevant for analysis are the voltage V, current I, and resistance R. *Voltage* across a circuit is the energy released when a unit of charge moves downhill from the higher potential to the lower. *Current* is defined as the flow of a charge through a circuit. *Resistance* is a property of an element of a circuit that opposes *current*. These quantities are related by *Ohm's law*: $V = IR$. German Physicist, Gustav Robert Kirchhoff, contributed to the fundamental understanding of electrical circuits while studying at the University of Königsberg in 1845. Kirchhoff's Laws capture simple circuit properties in the two statements: all the current flowing into a junction must flow out of it, and the sum of the products IR around a closed path is equal to the total voltage of the path. The relevant linear algebra tools are writing linear equations in matrix form and performing row operations to obtain reduced row echelon form.

Introduction To Circuit Analysis

In the modern-day, electronic circuits are abundant in every form of technology. An example of a simple circuit consists of a direct current battery, wires, a switch, and a light-emitting diode (LED). A circuit is complete when one wire connects the positive end of the battery to the positive terminal of the LED, and one wire connects the negative end of the battery to the negative terminal of the LED (see figure 1).

Two main quantities are necessary to keep track of in circuits; voltage represented by V and current represented by I. The *voltage* between two points is the cost in energy required to move a unit of positive charge from a lower potential to a higher potential. Equivalently, it is the energy released when a unit of

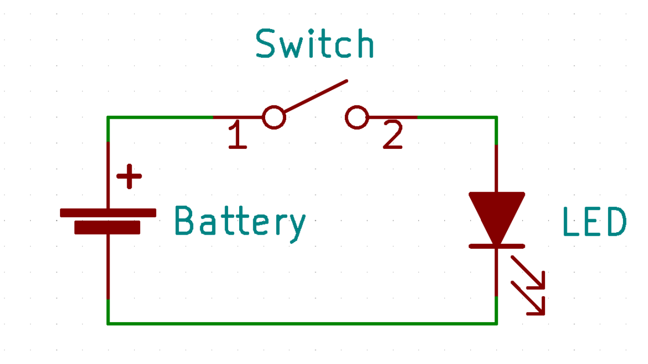

Figure 1: Simple LED circuit.

charge moves downhill from the higher potential to the lower [1]. A joule J of work is done in moving a coulomb C of charge across a potential difference of $1V$ (1 volt).

Current is the rate of flow of electric charge past a point. The unit of measure is denoted ampere, or amp A. A current of $1A$ is equal to the flow of 1 coulomb of charge per second $1\frac{C}{s}$. By convention, current in a circuit is considered to flow from a net high potential (positive) to a net low potential (negative) point [1].

As far as how currents and voltages are related in a given circuit, the simple concept is that voltages are what is needed to be applied across the circuit for current to flow through it. For the circuit in figure 1, when the switch is closed, current flows from the positive terminal of the battery, through the LED, performs useful work as it turns the light on and finds its way to the negative terminal of the battery. This clockwise flow of current that forms a closed loop, finishing where it starts, is an essential characteristic that allows us to make valuable inferences about how to analyze circuits.

Similar to the closed-loop formed by the current, there is an analogous loop that we can construct by thinking of the voltage drop across the circuit. To understand the voltage drop, we will first need to understand the LED's role in this circuit. We mentioned that useful work is performed by the current when it flows through the LED. This characteristic can also be described as a resistance (R) associated with the LED. Keeping in mind that the current flowing through the LED and, therefore, the entire circuit is proportional to the voltage across it, we can construct a quantity for the resistance of the LED.

$$R = V/I \tag{1}$$

This relationship is known as Ohm's Law, and for this reason, the units of resistance (R) are ohms (Ω). By rearranging the equation above, we can find what the voltage drop across the LED must be.

$$V = IR \tag{2}$$

From this equation, it is apparent that the voltage drop across this circuit is equal to the resistance of the LED multiplied by the current that flows through it.

An important side note: LEDs have terribly small resistance since they are incredibly efficient at producing light. This implies that if we use a standard 9V battery, we may provide the LED with much more current than it is designed for and destroy it. In a standard circuit of this type, there would be a small chunk of resistive material with a defined resistance R wired into the circuit (perhaps as in figure 2) such that the current flowing through the circuit is of the right magnitude for the LED to operate without breaking.

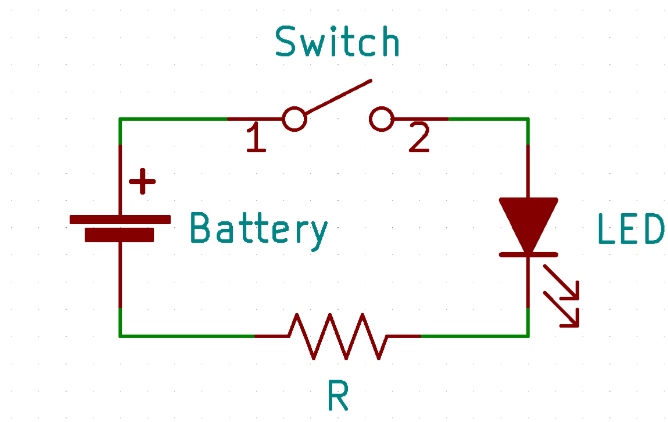

Figure 2: Simple LED circuit in series with a resistor.

Kirchhoff's Laws

Now we have defined everything we need to consider the laws that govern linear electronic circuits such as the example discussed here. These properties are known as Kirchhoff's Circuit Laws named after the German Physicist, Gustav Robert Kirchhoff. The two properties are as follows:

1. All the current flowing into a junction must flow out of it.
2. The sum of the products IR around a closed path is equal to the total voltage of the path.

1 History

Gustav Robert Kirchhoff, Son of Friedrich Kirchhoff and Johanna Henriette Wittke, was born in 1824 in the town of Königsberg, a port city of Prussia. Kirchhoff excelled in his studies and, in 1843, attended the Albertus University of Königsberg, where he studied mathematics and physics from two very prominent names: Carl Jacobi, who worked on developing differential calculus and is the namesake of the jacobian matrix, and Franz Neumann, worked on crystallography and electrical inductance. Neumann, in particular, had a significant influence on Kirchhoff and was the reason Kirchhoff began studying electrical circuits. While Kirchhoff and Neumann were working on circuits, Kirchhoff developed his rules for analyzing simple circuits, often referred to as Kirchhoff's laws. Kirchhoff's rules are an extension of ohm's law and appear in just about every introductory circuit analysis textbook because of how easy they are to understand and apply. Kirchhoff developed the rules by thinking of circuits as loops and nodes that can be quantified and solved as a system of linear equations. These laws were the source of what became his doctoral dissertation but were only the beginning of his working career.

In 1847 Kirchhoff graduated from Albertus University of Königsberg and moved to Berlin, where he

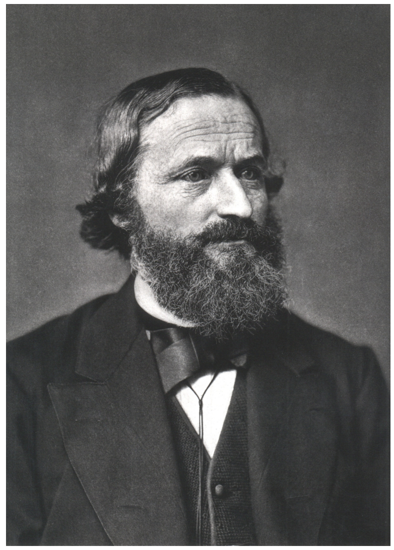

Figure 3: Gustav Robert Kirchhoff (1824-1887)

began teaching. Unfortunately for Kirchhoff, he moved to Berlin during a period of political unrest and taught unpaid for three years. The time he spent there was not entirely wasted, as he was able to correct a misconception that stemmed from accepted knowledge regarding electrical circuits and steady-state current. He then moved on in 1850 to Breslau, a city in Poland, and began formally teaching as an extraordinary professor. While there, he met Robert Bunsen, the namesake of the Bunsen burner, and the two became very good friends who would continue to work together for many years. In 1854, Bunsen encouraged Kirchhoff to go with him to teach at the University of Heidelberg, Germany as a professor of physics. Kirchhoff accepted and was given the position of professor in physics, where he and Bunsen began a very fruitful collaboration along with the likes of Hermann von Helmholtz.

Around this time, Kirchhoff was still working on his electrical circuits and current theories. He concluded that the current velocity in a wire was independent of the composition of the wire and was almost exactly equal to the velocity of light. A second group of scientists simultaneously came to this conclusion. However, it was not until James Clerk Maxwell that the connection was made that light is fundamentally an electromagnetic phenomenon. Similarly, at this time, Kirchhoff and Bunsen began working on what Kirchhoff would later coin the term black body radiation. Specifically, a fellow researcher, Joseph von Fraunhofer, had started to discover the phenomenon of emission lines that seemed to match the absorption lines present in the sun's spectrum. Kirchhoff made this fundamental breakthrough by producing purer forms of substances than had previously been able to be produced. He was then able to see, in 1859, that each element had a uniquely characteristic spectrum. He presented his law of radiation, stating that the emission and absorption lines appear at the same wavelength of light for a given atom or molecule.

Kirchhoff and Bunsen, armed with their new knowledge of emission lines, began to take a closer look at the sun. They were able to discover two new elements this way and understand what absorption lines are. Kirchhoff first proposed that emission lines stem from individual atoms in the diffuse atmosphere of the sun absorb specific wavelengths of light that correspond to particular interactions. This new understanding began a new era of astronomy and astrophysics.

During all of this, Kirchhoff met his wife Clara Richelot, and they married in 1857, with whom he had had three sons and two daughters. In 1869 Clara, unfortunately, passed away, leaving Kirchhoff to raise the children on his own. This transition was made more difficult for Kirchhoff as he had a disability that caused him to spend much of his life on crutches or in a wheelchair. He later married Luise Brömmel in 1872 and continued to teach at the University of Heidelberg until 1875, when Kirchhoff realized that his disability was making his research nearly impossible. He took up the position of the chair of mathematical physics at Berlin so that he could continue to write and teach. He stayed in Berlin until his eventual death in 1887.

2 Mathematics Background Information

Kirchhoff's laws are most useful when applied along with mathematical tools from the field of linear algebra. Circuits where Kirchhoff's laws are relevant can be constructed as linear equations. Thus the important mathematical tools in this context are: Expressing linear equations in an augmented matrix form and row reduction using Gauss-Jordan elimination to obtain reduced row echelon form.

2.1 How to put linear equations into matrix form

A system of linear equations can be represented in matrix form using a coefficient matrix, a variable matrix, and a constant matrix. A **coefficient matrix** is a matrix containing only the coefficients of the linear system. A **variable matrix** is a matrix is a matrix containing only the variables of the linear system. Finally the **constant matrix** is a matrix containing only the constant values of the linear system. When dealing with Kirchhoff's laws the coefficient matrix can be formed by aligning the coefficients of the variables of each equation in a row. Making sure that each equation is written in standard form with the constant terms on the right side of the equality. For an example we shall look at a simple system of linear equations.

$$x - y + z = 0 \tag{3}$$

$$3x + 2y = 7 \tag{4}$$

$$2y + 4z = 8 \tag{5}$$

We have a system of three linear equations with the variables of x, y, and z. We can now take this system of linear equations and put them into an augmented matrix. On the right side of the equations we have the constant terms. The three numbers in that order correspond to the first, second, and third equations, and will therefore take the places at the first, second, and third rows in the constant matrix. To put all of this into an **augmented matrix** form, all that needs to be done is to take your coefficient matrix and put it with your constant matrix. The coefficient matrix of our system is,

$$\begin{bmatrix} 1 & -1 & 1 \\ 3 & 2 & 0 \\ 0 & 2 & 4 \end{bmatrix} \tag{6}$$

and the corresponding constant matrix is,

$$\begin{bmatrix} 0 \\ 7 \\ 8 \end{bmatrix} \tag{7}$$

Therefore when you combine the coefficient matrix and the constant matrix, you will get the augmented matrix which turns out to be,

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 3 & 2 & 0 & 7 \\ 0 & 2 & 4 & 8 \end{bmatrix} \tag{8}$$

2.2 How to row reduce using Gauss-Jordan elimination

Once we have put our linear system of equations into an augmented matrix we can now start to solve this system. The necessary mathematics needed to solve the system is the ability to row reduce a matrix, and using Gauss-Jordan elimination to get the matrix into a reduced row echelon form. To **row reduce** a matrix you need to understand what row operations are first. **Row operations** are mathematical operations on the rows of a matrix representing a system of linear equations that result in a new matrix representing a new system of linear equations that has the same solution as the first. That includes interchanging two rows in the matrix, multiplying a row by a nonzero constant, and adding a multiple of a row to another row. On the other hand it is important to know what Gauss-Jordan elimination is in order to get your matrix into row reduced echelon form. **Gauss-Jordan elimination** is the process of applying the row operations as stated above to a matrix in order to put it into reduced row echelon form.

2.3 Reduced row echelon form

An augmented matrix is in **reduced row echelon form** if and only if the first nonzero entry in a row is the number one (we call this the leading entry). All zero rows appear at the bottom of the matrix. If a row has a leading entry, then all the rows below it have leading entries which lie in columns to the right. Lastly, every column that has a leading entry has zeros in every position above and below its leading one. When using Gauss Jordan elimination and row reducing our example above until we get it in reduced row echelon form we get the result,

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \tag{9}$$

This form gives us all the information needed to make a conclusion about or system. As a recap of the mathematics used to get here, we put a system of linear equations into an augmented matrix, then used Gauss Jordan elimination by applying row operations to the matrix to row reduce it, and finally continuing to row reduce until we reached reduced row echelon form.

3 Linear Algebra Application Example

To better understand how to apply linear algebra techniques to solve Kirchhoff's circuit rules, we will use a simple circuit to demonstrate all of the steps before moving on to a more complicated circuit. This accomplishes two goals; the first is to show all of the necessary techniques in the context of a circuit, and the second is to show that a larger circuit can be just as easy to solve as a simple one. consider the circuit shown in figure 4.

By applying Kirchhoff's rules, we see two loops and two junctions. However, in the case of this circuit, the two junctions contain the same information, so we only need to look at one of them. We will be able to construct three equations from these rules, meaning we can solve for three unknowns. We will find the three currents that pass through the three resistors respectively for this circuit. We will construct the loops to say the left half is flowing counterclockwise and the right half is flowing clockwise. This allows us to make the first two equations which can be found to be $5 - 5I_1 - 10I_3 = 0$ and $10 - 20I_2 - 10I_3 = 0$. these are not yet in the proper form for a linear equation, but we arrive at two linear equations by moving a few terms around and distributing negative signs properly. Additionally, even though R2 does not appear in the first loop or R1 in the second, we will include their components in the equations to organize the equation for a later step. The equations are as follows:

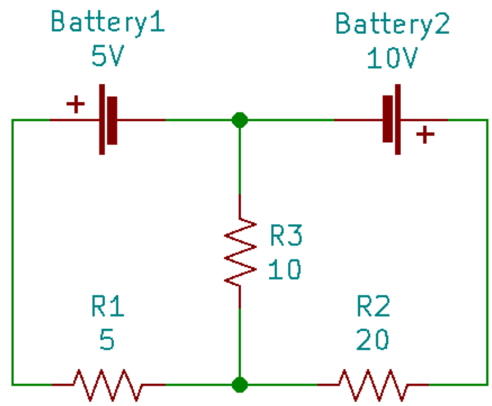

Figure 4: A simple example circuit

1. $5I_1 + 0I_2 + 10I_3 = 5$
2. $0I_1 + 20I_2 + 10I_3 = 10$

The junction rule is straightforward in this circuit because even though there are two junctions, they both contain the same information. Referring to the directions we define in the loops, we find the current from R1 and R2 both flow into the junction at the bottom of the circuit, meaning their sum must be equal to the current through R3. Equivalently, we find the current moving up through R3 splits left and right, so the sum of the currents through R2 and R1 must be the same as the current through R3. this can be written algebraically as $I_1 + I_2 = I_3$. once again, this is not yet in the form necessary to be classified as a linear equation, but simply manipulations can get up to the necessary equations.

3. $I_1 + I_2 - I_3 = 0$

We now have a system of three equations and three unknowns, so we may begin to solve the system. The first step is to rewrite the system in matrix form. We will construct a coefficient matrix, a variable matrix, and a constant matrix straight from the three equations above. The zeros written into equations 1 and 2 will be helpful here for writing the coefficient matrix because the ordering is much easier to obtain. Therefore, the system of linear equations above can be rewritten in matrix form as follows.

$$\begin{bmatrix} 5 & 0 & 10 \\ 0 & 20 & 10 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 0 \end{bmatrix} \tag{10}$$

Using this information, we can then construct what is referred to as an augmented matrix. We will take the coefficient matrix and the constant matrix and join them together into a single, larger, augmented matrix. the augmented matrix will look like

$$\begin{bmatrix} 5 & 0 & 10 & 5 \\ 0 & 20 & 10 & 10 \\ 1 & 1 & -1 & 0 \end{bmatrix}$$

Finally, we may begin solving the linear equation system using the Gauss-Jordan elimination method, which leaves the matrix in reduced row-echelon form. This step is usually done in software like wolfram Mathematica or online for more complicated matrices, but a three-by-four matrix can easily be solved by hand. The first step we take will be to get the leading ones in the rows. This step is made easier because the second row already has its first entry as 0. We will divide the first row by five and the second row by 20 to get to the state

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 1 & 1 & -1 & 0 \end{bmatrix}$$

Next, we will subtract the first row from the third, and after that, we will subtract the second row from the third. To finish out the third row, we will multiply the entire row by the reciprocal of the third entry to get a leading 1. this reciprocal turns out to be $\frac{-2}{7}$.

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{3}{7} \end{bmatrix}$$

The final steps are to remove the two from the first row and the first $\frac{1}{2}$ from the second row. We will do this by using multiples of the third row, specifically $2 * Row_3 + Row_1$ and $\frac{1}{2} * Row_3 + Row_2$. This finally yields us the reduced row-echelon form of the augmented matrix, giving us the solution to the system.

$$\begin{bmatrix} 1 & 0 & 0 & \frac{1}{7} \\ 0 & 1 & 0 & \frac{2}{7} \\ 0 & 0 & 1 & \frac{3}{7} \end{bmatrix}$$

This tells us that the current passing through resistor 1 is $\frac{1}{7}^{th}$ of an amp, and the current passing through resistor 2 is $\frac{2}{7}^{ths}$ of an amp, and the current passing through resistor 3 is $\frac{3}{7}^{ths}$ of an amp. in other words, $I_1 = \frac{1}{7}$, $I_2 = \frac{2}{7}$, and $I_3 = \frac{3}{7}$. Now that we know what it takes to solve a simple circuit using Kirchhoff's rules, we will explore a more difficult-looking circuit and find that the core principles will remain the same.

4 Solution of Larger Example

We will let our new, more difficult circuit to analyze be the circuit shown in figure 5. Note here that there are a series of arrows on the diagram to show the direction of current flow and number labels within the blocks to help enumerate the loops used in Kirchhoff's rules. We will use the same process as outlined above and begin by creating an equation for each loop in the circuit, of which there will be 5. the equations in simplified terms are as follows:

1. $2I_4 + 4I_6 + 12I_7 + 10I_8 = 15$
2. $10I_8 - 8I_9 + 16I_10 = 10$
3. $18I_2 - 6I_3 + 2I_4 = 5$
4. $10I_1 + 6I_3 + 4I_6 = 20$
5. $14I_5 + 12I_7 + 8I_9 = 15$

Next are the 6 junction rules. However, only five are needed since there are ten unknowns, and we already have five equations. The junction in figure 5 are denoted by large green dots, and we will use them starting top left, left to right, and top to bottom. After moving all of the terms to one side, we are left with the following five linear equations being:

6. $-I_1 + I_2 + I_3 = 0$
7. $I_3 + I_4 - I_6 = 0$
8. $I_1 - I_6 + I_8 - I_{10} = 0$
9. $-I_2 + I_4 + I_5 - I_7 = 0$
10. $-I_7 + I_8 + I_9 = 0$

We may then compile all of these ten equations and ten unknowns into an augmented matrix of size ten-by-eleven, shown below.

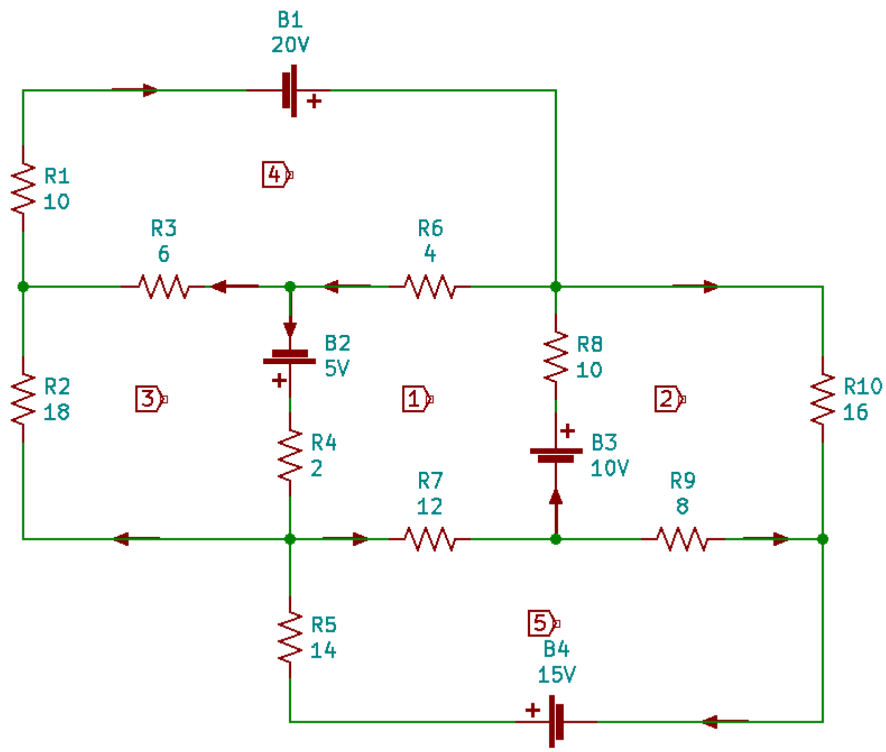

Figure 5: Complex circuit

$$\begin{bmatrix} 0 & 0 & 0 & 2 & 0 & 4 & 12 & 10 & 0 & 0 & 15 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & -8 & 16 & 10 \\ 0 & 18 & -6 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 5 \\ 10 & 0 & 6 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 20 \\ 0 & 0 & 0 & 0 & 14 & 0 & 12 & 0 & 8 & 0 & 15 \\ -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & 0 \end{bmatrix} \tag{11}$$

Since the matrix is too large to solve by hand, we used Wolfram Mathematica to compute the reduced row-echelon form of the augmented matrix. The result is shown below. The ugly-looking fractions on the right-hand columns correspond to the currents passing through the ten resisters. Looking at these numbers in decimal form shows all of the currents are somewhere between 1.2 and 0.12 amps, which follows what intuition would say.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 331535/287743 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 133305/287743 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 198230/287743 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 228605/575486 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 327915/575486 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 625065/575486 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 144955/287743 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 109795/287743 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 35160/287743 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 257595/575486 \end{bmatrix} \quad (12)$$

The current passing through each of the resistors is as follows, $I_1 = 1.15A$, $I_2 = 0.46A$, $I_3 = 0.69A$, $I_4 = 0.40A$, $I_5 = 0.57A$, $I_6 = 1.09A$, $I_7 = 0.50A$, $I_8 = 0.38A$, $I_9 = 0.12A$, and $I_{10} = 0.45A$.

Even though the scale of this circuit is significantly larger than the last, and there are over three times as many equation, all of the principles are the same. We took the circuit one loop at a time and made a linear equation for each. Then we took the circuit one junction at a time and ensured current in equals current out for each. We then set up an augmented matrix and row reduced it to get an answer. The only added difficulty of the large matrix is the number of variables to keep track of, but the problem of analysis remains the same.

5 Conclusion

For any linear circuit such as one consisting of various resistors and batteries, Kirchhoff's laws can be applies to make inferences about the voltage and currents' behaviors. Kirchhoff's laws state that all the current flowing into a junction must flow out of it and the sum of the products IR around a closed path is equal to the total voltage of the path. Once inferences about the circuit are made, linear algebra tools such as matrix representation of equations and Gauss-Jordan elimination can be used to solve for values of interest. The common value of interest in a circuit of various resistors and batteries is the current through certain branches of the circuit.

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